A Heuristic Algorithm for Two-Dimensional Rectangular Packing Problems

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Abstract

Two-dimensional rectangular packing problems occur frequently in various industries, such as paper, wood, glass, and steel. This paper focuses on the two-dimensional open dimension rectangular packing problem with the intention of placing a set of small rectangular items without overlapping into a large rectangle of minimal area. This problem is known as a NP-hard problem and conventional integer programming techniques may take exponential time to deterministically solve the problem to global optimality. This paper proposes an efficient heuristic algorithm to obtain a feasible solution to two-dimensional open dimension rectangular packing problems. Compared with the deterministic methods, the reduced solution time enhances the applicability of the proposed heuristic method.

Keywords: Two-dimensional rectangular packing, open dimension problems, heuristic algorithm

1. Introduction

The rectangular packing problem discussed in this study places a set of small rectangular items without overlapping into a large rectangle in such a way that the area of the enveloping rectangle is minimized. This problem arises frequently in different raw material manufacturing industries. In manufacturing industries, material utilization is an important issue. A high level of material utilization is of particular interest to mass-production manufacturing industries and small improvements in the layout can result in large savings in raw materials, significantly reducing production costs. In addition to manufacturing industries, this general problem has several practical applications, for instance, allocating the location of departments in the minimum area, and allocating all incoming ships to berths with the minimum completion time. Lodi et al. (2002a, 2002b) provided a complete survey of the applications, theoretical results and algorithmic advances in two-dimensional packing problems during 2000. Wäscher et al. (2007) categorized the literature on cutting and packing problems from the years between 1995 and 2004. According to the Wäscher et al. (2007) classification, the rectangular packing problem addressed in this study belongs to the category of two-dimensional open dimension problems.

Many approaches to two-dimensional rectangular packing problems have been proposed and can be classified into deterministic and heuristic. Although the deterministic method can guarantee to obtain a global optimum for the two-dimensional rectangular packing problem, the NP-hard characteristic of the problem implies that the solution time required by the deterministic optimization techniques may increase exponentially to gain the global optimal solution. Due to the complexity of the problems, most research on these topics is based on heuristic methods. Jakobs (1996) applied genetic algorithms to the two-dimensional packing problem and employed the bottom-left strategy to locate the small rectangles. Hopper and Turton (2001) combined a meta-heuristic algorithm (genetic algorithm, simulated annealing, and naïve evolution) and a heuristic packing routine to place the small items on the large rectangle. Leung et al. (2003) applied a mixed, simulated annealing-genetic algorithm to two-dimensional orthogonal packing problems. Lin (2006) proposed a genetic algorithm using a novel random packing process and an encoding scheme for solving two-dimensional assortment problems. Gonçalves (2007) proposed an algorithm hybridizing a placement procedure with a genetic algorithm based on random keys. Wei et al. (2011) proposed a greedy heuristic method that places each rectangle according to an evaluation function involving several components of the nature of the two-dimensional packing problems. Leung et al. (2012) applied a constructive heuristic algorithm, based on the fitness strategy, to generate a solution for a rectangular knapsack packing problem. By combining the greedy strategy and a simulated annealing algorithm, a hybrid heuristic algorithm was proposed to find an improved solution. Zhou et al. (2012) adopted the approach of bottom left corner occupation as the placement strategy, and then developed a hybrid framework combining a genetic algorithm and a tabu search to solve a two-dimensional bin packing problem. Blum and Schmid (2013) proposed an algorithm to tackle a 2D bin packing problem using an evolutionary algorithm making heavy use of a randomized one-pass heuristic for construction solutions. Kierkosz and Luczak (2014) presented a hybrid evolutionary algorithm for a two-dimensional non-guillotine packing problem. A tree search improvement procedure was then used with an initial solution obtained from the best solution obtained by the evolutionary algorithm. Lu et al. (2013) proposed an integrated algorithm incorporating a genetic algorithm, a corner arrangement method, and a production plan model to solve the cutting stock problems in the TFT-LCD industry. The heuristic algorithms mentioned above have been developed based on different methodologies and presented to solve two-dimensional rectangular packing problems. Although the heuristic methods cannot guarantee to obtain a global optimum, they have the advantage of easy implementation and offer better potential for complex problems. Table 1 summarizes the characteristics of different heuristic approaches as reviewed previously.

Most of the heuristic methods investigated the two-dimensional rectangular packing problem whereby the extension of the enveloping rectangle is fixed in both dimensions. This study proposes a heuristic algorithm for two-dimensional packing problems where both the length and the width are variable. The concept uses the remaining space of the existing enveloping rectangle, and balances the length and the width of the enveloping rectangle, if expansion is necessary, to enable the insertion of a new small rectangle. Section 2 describes the proposed heuristic algorithm. Section 3 demonstrates the allocation process of the proposed heuristic algorithm using numerical examples. Finally, several concluding remarks are discussed in Section 4.

Table 1: Characteristics of Different Heuristic Approaches

Approach	Methodology	Characteristics
Jakobs (1996)	Genetic algorithm	Place a set of polygons on a rectangular board with minimal height.
Hopper and Turton (2001)	Genetic algorithm, simulated annealing, naïve evolution.	Pack a set of rectangles onto a rectangular object with minimal used object space.
Leung et al. (2003)	Genetic algorithm, mixed simulated annealing-genetic algorithm	Pack rectangular pieces of predetermined sizes onto a large rectangular plate with minimal unused area (trim loss).
Lin (2006)	Genetic algorithm using a problem-specific encoding scheme and a novel packing process	Place a set of rectangles within another rectangle with a minimal area.
Gonçalves (2007)	Genetic algorithm	Place a set of small rectangles onto a larger stock rectangle with minimal trim loss.
Wei et al. (2011)	Greedy heuristic method	Pack rectangles of predetermined sizes into a large rectangular plate with the maximal packed area.
Lu et al. (2013)	Genetic algorithm	Find all possible cutting patterns for the cutting of various TFT-LCD plates of predetermined sizes from a glass substrate with minimal trim-loss.

2. Proposed Heuristic Method

The concept of the proposed algorithm is to balance the length and the width of the enveloping rectangle if expansion is necessary to enable the insertion of a new small rectangular item. Table 2 lists several terminologies used in the proposed heuristic algorithm.

The following describes the detailed steps of the proposed algorithm for two-dimensional open dimension rectangular packing problems.

Step 1: Sort the small rectangles using the sum of their length and width to obtain an ordered set {$s\_rectangle_1$, $s\_rectangle_2$,…, $s\_rectangle_n$ } where $(p_1+q_1)\geq(p_2+q_2)\geq\cdots\geq(p_n+q_n)$.

Table 2: Terminologies of the Proposed Algorithm

Terminology	Meaning
n	The number of small rectangles that are required to be packed
$[p_i, q_i]$	The length (long side) p_i and width (short side) q_i of a small rectangle i.
$R_i=[R_{ix}, R_{iy}, s_i]$	The bottom-left coordinate (R_{ix}, R_{iy}) of a small rectangle i. s_i=0 if the short side q_i is parallel to the x-axis; otherwise, s_i=1 if the short side q_i is parallel to the y-axis.
J	The number of the remaining rectangular spaces.
$C_j=[C_{jx}, C_{jy}, C_{jw}, C_{jh}]$	The bottom-left coordinate (C_{jx}, C_{jy}) of the remaining rectangular space j, the length C_{jw} of the side parallel to the x-axis, and the length C_{jh} of the side parallel to the y-axis.
$M=[m_x, m_y]$	The top-right coordinate (m_x, m_y) of the enveloping rectangle.

Step 2: Let $i=1$. Rectangle i is put with the bottom-left corner aligned with the origin and the short side parallel to the x-axis. $R_i=[R_{ix}, R_{iy}, s_i]=[0,0,0]$, $M=[m_x, m_y]=[q_i, p_i]$. Let $J=0$.

Step 3: Let $i=i+1$. If $i>n$, then go to Step 7. If $J=0$, then go to Step 4; otherwise, let $j=1$ and go to Step 5.

Step 4: If $m_x \leq m_y$, then rectangle i is inserted with the short side parallel to the x-axis. $R_i=[R_{ix},R_{iy},s_i]=[m_x,0,0]$. Determine the required enveloping rectangle and the remaining rectangular spaces according to the following three conditions:

If $p_i<m_y$, then let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[m_x,p_i,q_i,m_y-p_i]$, $M=[m_x,m_y]=[m_x+q_i,m_y]$;

If $p_i=m_y$, then $M=[m_x,m_y]=[m_x+q_i,m_y]$;

If $p_i>m_y$, then let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[0,m_y,m_x,p_i-m_y]$, $M=[m_x,m_y]=[m_x+q_i,p_i]$.

If $m_x>m_y$, then rectangle i is inserted with the short side parallel to the y-axis. $R_i=[R_{ix},R_{iy},s_i]=[0,m_y,1]$. Determine the required enveloping rectangle and the remaining rectangular spaces according to the following three conditions:

If $p_i<m_x$, then let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[p_i,m_y,m_x-p_i,q_i]$, $M=[m_x,m_y]=[m_x,m_y+q_i]$;

If $p_i=m_x$, then $M=[m_x,m_y]=[m_x,m_y+q_i]$;

If $p_i>m_x$, then let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[m_x,0,p_i-m_x,m_y]$, $M=[m_x,m_y]=[p_i,m_y+q_i]$.

Go to Step 6.

Step 5: If $q_i\le C_{jw}$, then $M=[m_x,m_y]$, $R_i=[R_{ix},R_{iy},s_i]=[C_{jx},C_{jy},0]$. Determine the remaining rectangular spaces according to the following four conditions:

If $q_i<C_{jw}$ and $p_i<C_{jh}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx},C_{jy}+p_i,C_{jw},C_{jh}-p_i]$, let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[C_{jx}+q_i,C_{jy},C_{jw}-q_i,p_i]$;

If $q_i<C_{jw}$ and $p_i=C_{jh}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx}+q_i,C_{jy},C_{jw}-q_i,C_{jh}]$;

If $q_i=C_{jw}$ and $p_i<C_{jh}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx},C_{jy}+p_i,C_{jw},C_{jh}-p_i]$;

If $q_i=C_{jw}$ and $p_i=C_{jh}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[0,0,0,0]$.

Go to Step 6.

If $q_i\le C_{jh}$, then $M=[m_x,m_y]$ and $R_i=[R_{ix},R_{iy},s_i]=[C_{jx},C_{jy},1]$. Determine the remaining rectangular spaces according to the following four conditions:

If $q_i<C_{jh}$ and $p_i<C_{jw}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx},C_{jy}+q_i,C_{jw},C_{jh}-q_i]$, let $J=J+1$, $C_J=[C_{Jx},C_{Jy},C_{Jw},C_{Jh}]=[C_{jx}+p_i,C_{jy},C_{jw}-p_i,q_i]$;

If $q_i<C_{jh}$ and $p_i=C_{jw}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx},C_{jy}+q_i,C_{jw},C_{jh}-q_i]$;

If $q_i=C_{jh}$ and $p_i<C_{jw}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[C_{jx}+p_i,C_{jy},C_{jw}-p_i,C_{jh}]$;

If $q_i=C_{jh}$ and $p_i=C_{jw}$, then $C_j=[C_{jx},C_{jy},C_{jw},C_{jh}]=[0,0,0,0]$.

Go to Step 6.

Let $j=j+1$. If $j\le J$, then reiterate Step 5; otherwise, go to Step 4.

Step 6: Remove the remaining rectangular space that is fully filled with the small rectangles. Re-sort the remaining rectangular spaces by the area of each remaining rectangular space. Go to Step 3.

Step 7: Output $M=[m_x,m_y]$, $R_i=[R_{ix},R_{iy},s_i]$, $i=1,2,\ldots,n$.

3. Numerical Examples

Herein is an example with eight small rectangles required to be allocated without overlapping into a large rectangle to demonstrate the placement process of the proposed heuristic algorithm. The size of the eight small rectangles are (50,15), (38,15), (35,17), (30,11), (26,20), (25,15), (24,12), (18,4), respectively. Figures 1 to 8 illustrate the allocation of the eight small rectangles sequentially according to the proposed algorithm.

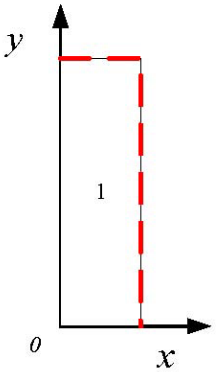

Figure 1: The Placement of Rectangle 1

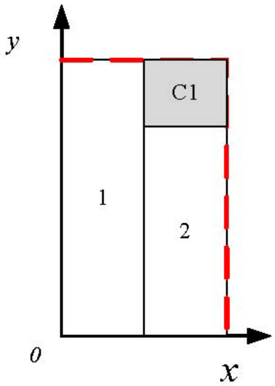

Figure 2: The Placement of Rectangle 2 and the Remaining Spaces

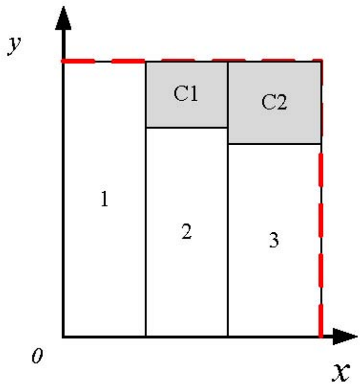

Figure 3: The Placement of Rectangle 3 and the Remaining Spaces

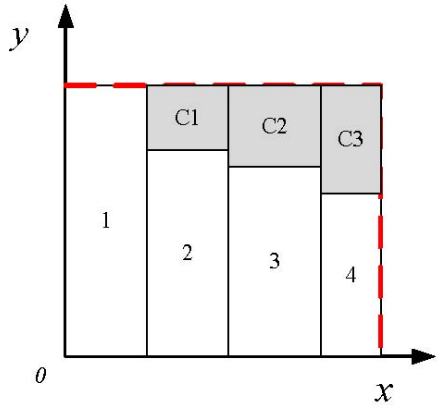

Figure 4: The Placement of Rectangle 4 and the Remaining Spaces

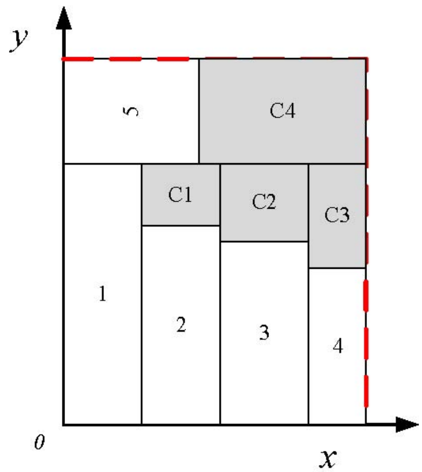

Figure 5: The Placement of Rectangle 5 and the Remaining Spaces

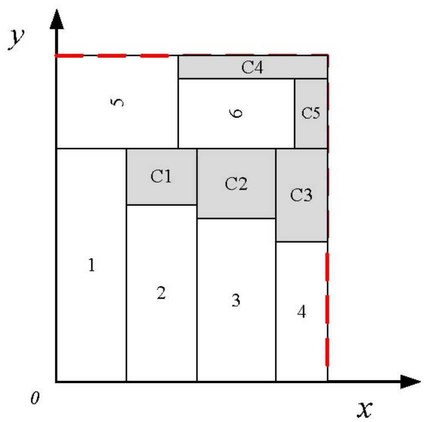

Figure 6: The Placement of Rectangle 6 and the Remaining Spaces

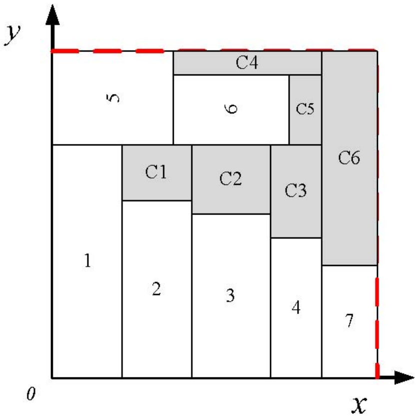

Figure 7: The Placement of Rectangle 7 and the Remaining Spaces

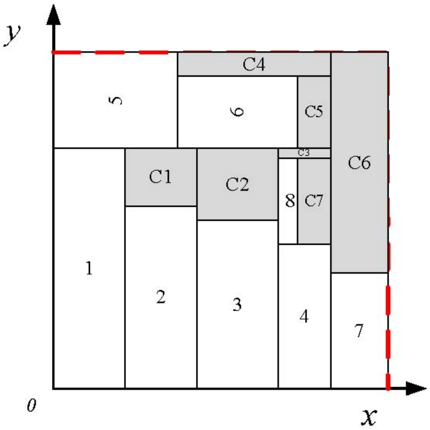

Figure 8: The Placement of Rectangle 8 and the Remaining Spaces

Tsai et al. (2013) proposed a deterministic approach to solve two-dimensional rectangular packing problems. Their method utilized a simple heuristic method to find the feasible area of the enveloping rectangle as follows:

$$\min\left((\sum_{i=1}^{n} p_i)(\max_i q_i),(\sum_{i=1}^{n} q_i)(\max_i p_i)\right)$$

To show the solution quality of the proposed algorithm, four problems (Problems 1 to 4) drawn from Tsai et al. (2013) are solved to compare the above heuristic method with the proposed algorithm. So as to treat the problem with additional small rectangles, problems 5 and 6 are extensions of problem 4. The sizes of the rectangles to be allocated in each problem are listed as follows:

Problem 1 with 5 rectangles: (33,10), (30,11), (25,15), (18,14), (18,10).

Problem 2 with 6 rectangles: (40,18), (36,12), (32,24), (25,20), (20,16), (20,13).

Problem 3 with 7 rectangles: (45,10), (40,18), (32,24), (25,5), (21,12), (20,16), (13,5).

Problem 4 with 8 rectangles: (50,15), (38,15), (35,17), (30,11), (26,20), (25,15), (24,12), (18,4).

Problem 5 with 9 rectangles: (50,15), (38,15), (35,17), (30,11), (26,20), (25,15), (24,12), (18,4), (15,12).

Problem 6 with 10 rectangles: (50,15), (38,15), (35,17), (30,11), (26,20), (25,15), (24,12), (18,4), (15,12), (10,8).

Table 3 lists the area of the enveloping rectangle solved by the heuristic method and the proposed algorithm. The proposed algorithm is able to obtain a smaller enveloping rectangle than the heuristic method in all problems except Problem 3. Additionally, the difference is more significant as the number of small rectangles increases.

Table 3: Comparison of the Area of the Enveloping Rectangle

Problem	**Heuristic Method**	**Proposed Algorithm**
Problem 1	1860	1692
Problem 2	4120	4020
Problem 3	4050	4402
Problem 4	4920	4900
Problem 5	5220	4900
Problem 6	5420	4900

4. Conclusions and Future Work

This study proposes a heuristic algorithm to solve two-dimensional open dimension rectangular packing problems. Since the NP-hard characteristic of the two-dimensional open dimension rectangular packing problem means the solution time required by the deterministic optimization techniques increases significantly, developing an efficient heuristic algorithm is more appropriate for treating large-scale problems in real applications. In this study, we propose a novel algorithm that balances the length and the width of the enveloping rectangle when allocating placing a set of small rectangular items without overlapping.

In the future, the proposed algorithm can be modified by considering the combination of the remaining spaces. Since the area of the remaining space is the area of the enveloping rectangle subtracted from the total area of the given set of rectangular pieces, the minimal area of the enveloping rectangle results in the minimal area of the remaining spaces. However, the small rectangular items should be assigned across adjacent remaining spaces in the placement process. Additionally, the impact of placing the small rectangles with different sequences can be investigated for a possible improvement in the proposed algorithm. Finally, the solution obtained from the proposed algorithm may be further integrated into the deterministic optimization technique for two-dimensional open dimension rectangular packing problems. In the deterministic mathematical programming model, adding an inequality constraint that forces the minimal area of the enveloping rectangle to be less than the area of the enveloping rectangle found by the proposed algorithm may reduce the solution time of the deterministic model. Further experiments should be conducted to verify the effects of integrating the results from the proposed method into the deterministic technique so as to consider the solution efficiency and solution quality simultaneously.

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